Time-Varying Factor Graphical Models

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Summary

1. **Goal**: construct time-varying financial portfolio under structural breaks;
2. **Contributions**:
   - Develop time-varying inverse covariance (precision) matrix estimator;
   - Integrate time-series graphical modelling and latent variable network inference;
   - Demonstrate superior performance of TVFGL using empirical application to the components of the S&P500 components during the first wave of COVID-19.

Time-Varying Network Inference

- Relationships between stocks evolve over time;
- Portfolio weight vector is a function of precision matrix;
- Assumption of constant precision matrix is not realistic

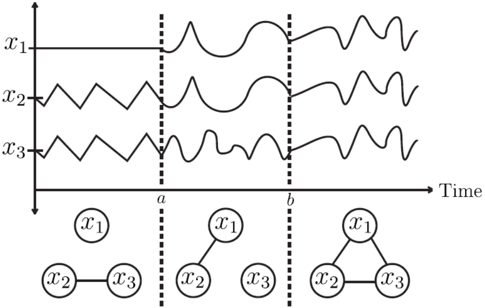

Figure: A dynamic network with associated time series readings. *Source*: Hallac et al., KDD 2017

Stock Returns R are driven by Common Factors F

$$\underbrace{\mathbf{R}}_{p\times N} = \underbrace{\mathbf{B}}_{p\times K}\mathbf{F} + \mathbf{E} \tag{1}$$

Population quantities:

$$\boldsymbol{\Sigma}_\varepsilon = N^{-1}\mathbf{E}\mathbf{E}'; \qquad \boldsymbol{\Theta}_\varepsilon = \boldsymbol{\Sigma}_\varepsilon^{-1},$$
$$\boldsymbol{\Sigma}_f = N^{-1}\mathbf{F}\mathbf{F}'; \qquad \boldsymbol{\Theta}_f = \boldsymbol{\Sigma}_f^{-1},$$
$$\boldsymbol{\Sigma} = N^{-1}\mathbf{R}\mathbf{R}'; \qquad \boldsymbol{\Theta} = \boldsymbol{\Sigma}^{-1}$$

Sample counterparts:

$$\widehat{\boldsymbol{\Sigma}}_\varepsilon = N^{-1}\widehat{\mathbf{E}}\widehat{\mathbf{E}}'; \qquad \widehat{\boldsymbol{\Theta}}_\varepsilon \leftarrow \text{GL(Weighted Graphical Lasso)},$$
$$\widehat{\boldsymbol{\Sigma}}_f = N^{-1}\widehat{\mathbf{F}}\widehat{\mathbf{F}}'; \qquad \widehat{\boldsymbol{\Theta}}_f = \widehat{\boldsymbol{\Sigma}}_f^{-1}$$

Goal: estimate $\boldsymbol{\Theta}$ to get portfolio weights $\mathbf{w} = f(\boldsymbol{\Theta})$.

Time-Varying Factor Graphical Lasso (TVFGL)

- Model the change in precision matrix of stock returns $\boldsymbol{\Theta}$ due to T known structural breaks t_i, $i = 1, \ldots, T$
- Use the Sherman-Morrison-Woodbury (SMW) formula:

$$\text{TVFGL} \rightarrow \widehat{\boldsymbol{\Theta}}_i = \underbrace{\widehat{\boldsymbol{\Theta}}_{\varepsilon,i}}_{\text{GL}} - \widehat{\boldsymbol{\Theta}}_{\varepsilon,i}\widehat{\mathbf{B}}[\underbrace{\widehat{\boldsymbol{\Theta}}_f}_{\text{FM}} + \widehat{\mathbf{B}}'\widehat{\boldsymbol{\Theta}}_{\varepsilon,i}\widehat{\mathbf{B}}]^{-1}\underbrace{\widehat{\mathbf{B}}'}_{\text{FM}}\widehat{\boldsymbol{\Theta}}_{\varepsilon,i} \quad (2)$$

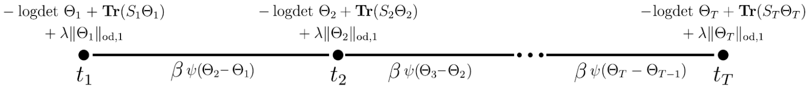

Figure: Change of precision matrix over time with β being the penalty that enforces temporal consistency and ψ being a convex penalty function. *Source*: Hallac et al., KDD 2017

- Use $\widehat{\boldsymbol{\Theta}}_i$ to get portfolio weights $\widehat{\mathbf{w}}_i = f(\widehat{\boldsymbol{\Theta}}_i)$.

TVFGL: Tuning and Solution Method

- **Smoothing functions candidates**: Lasso ($\psi = \sum_{j,k} |\cdot|$); Group Lasso ($\psi = \sum_k \|\cdot_k\|_2$); Laplacian ($\psi = \sum_{j,k} (\cdot_{jk})^2$); Max norm penalty ($\psi = \sum_k \max_j |\cdot_{jk}|$).
- **Solution Method**: ADMM.

$$
\begin{aligned}
\min_{\boldsymbol{\Theta}_i \succ 0, \forall i} \sum_{i=1}^{T} n_i \Big[\operatorname{trace}\Big(\widehat{\boldsymbol{\Sigma}}_i \boldsymbol{\Theta}_i \Big) - \log\det \boldsymbol{\Theta}_i \Big] + \lambda \|\boldsymbol{\Theta}_i\|_{od,1} \\
+ \beta \sum_{i=2}^{T} \psi(\boldsymbol{\Theta}_i - \boldsymbol{\Theta}_{i-1}) \\
\text{s.t. } \mathbf{Z}_{i,0} = \boldsymbol{\Theta}_i, \ \text{for } i = 1, \ldots, T \\
\Big(\mathbf{Z}_{i-1,1}, \mathbf{Z}_{i,2} \Big) = \Big(\boldsymbol{\Theta}_{i-1}, \boldsymbol{\Theta}_i \Big), \ \text{for } i = 2, \ldots, T.
\end{aligned}
$$

- Let $\mathbf{A}^k = \dfrac{\mathbf{Z}_{i,0}^k + \mathbf{Z}_{i-1,1}^k + \mathbf{Z}_{i,2}^k - \mathbf{U}_{i,0}^k - \mathbf{U}_{i-1,1}^k - \mathbf{U}_{i,2}^k}{3}$;
- Let $\mathbf{Q}_i \mathbf{\Lambda}_i \mathbf{Q}_i'$ be the eigendecomposition of $\frac{1}{\eta}\mathbf{A}^k - \widehat{\mathbf{\Sigma}}_i$, where $\eta = \frac{n_i}{3\rho}$;
- Closed-form solution:

$$\mathbf{\Theta}_i = \frac{n_i}{6\rho}\mathbf{Q}_i\left(\mathbf{\Lambda}_i + \sqrt{\mathbf{\Lambda}_i^2 + \frac{12\rho}{n_i}\mathbf{I}}\right)\mathbf{Q}_i', \tag{3}$$

where ρ is the augmented Lagrangian multiplier

Data

Data: Daily returns of the components of the S&P500:

- Full sample: 1089 observations on 500 stocks from January 3, 2017 - April 30, 2021.

Figure: S&P500 Index. *Source*: CNBC

- Breaks: February 10, 2020 and March 16, 2020.

Data

- Training: January 3, 2017 - December 31, 2020 (1007 obs).
- Test: January 1, 2021 - April 30, 2021 (82 obs).
- **Factors:** statistical factors (PC).
- **Targets:**
  (return target, risk target) = $(\mu, \sigma) = (0.0378\%, 0.013)$.
- **Tuning:** Use the first 2/3 of training data to estimate weights and jointly tune λ and β in the remaining 1/3 to yield the highest Sharpe Ratio.

Results

- FGL – constant precision matrix;
- FGL-postbreak – using observations after 3/16/20;
- TVFGL (1 break) – break at 3/16/20;
- TVFGL (2 breaks) – breaks at 2/10/20 and 3/16/20.

	Global Minimum Variance Portfolio			Markowitz Portfolio		
	Mean	**Risk**	**SR**	**Mean**	**Risk**	**SR**
FGL	0.0006	0.0056	0.1155	0.0006	0.0056	0.1068
FGL-postbreak	0.0009	0.0062	0.1390	0.0009	0.0063	0.1368
TVFGL (1 break)	0.0019	0.0133	0.1441	0.0021	0.0156	0.1378
TVFGL (2 breaks)	0.0049	0.0221	0.2217	0.0027	0.0141	0.1909

Table: Daily portfolio returns, risk and Sharpe ratio, $(\mu, \sigma) = (0.0378\%, 0.013)$.

Conclusions

- We develop a framework to estimate time-varying precision matrix under structural breaks;
- We derive a closed-form solution for precision matrix using ADMM;
- We demonstrate that under strong structural breaks relaxing the assumption of constant precision matrix improves portfolio return and Sharpe Ratio

Thank you!